



Fecho dos Morros, literally *Closing of the Hills*, a substantial tectonic uplift feature (≈ 40 m) responsible for the existence of the mild land-surface slope ($S_o \approx 0.00001$) and associated **runoff diffusion** in the Pantanal of Mato Grosso, Mato Grosso do Sul, Brazil.

IS THE DYNAMIC HYDRAULIC DIFFUSIVITY ALTOGETHER BETTER THAN ITS KINEMATIC COUNTERPART?

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ABSTRACT. A comparison between kinematic and dynamic hydraulic diffusivities using an actual catchment model of overland flow is accomplished. The kinematic diffusivity is the well known Hayami diffusivity. On the other hand, the dynamic diffusivity is a function of the Vedernikov number. Several examples of rainfall-runoff conversions using the catchment model show that the difference between these two formulations of hydraulic diffusivity is negligible. This may be attributed to the very low Vedernikov numbers typically featured in catchment models, or to the specification of kinematic and diffusion waves in the test cases, since the latter are the only ones likely to be encountered in actual practice. The question of the true nature of mixed kinematic-dynamic waves is examined using available analytical data. It is concluded that given their extremely strong dissipative tendencies, mixed waves may not be there for us to calculate. Nevertheless, the dynamic hydraulic diffusivity is advocated as the method of choice, because it is

the complete solution, it is applicable to all types of routing, and it does not significantly complicate the methodology.

1. INTRODUCTION

Runoff diffusion is the property of free-surface flows to diffuse appreciably, attenuating unsteady flow stages in Nature's apparently determined quest to return the flow to its steady uniform condition. Diffusion is characterized by the coefficient of the second-order term of the partial differential equation governing the flow (**Ponce, 2014a: Diffusion waves**). This coefficient is commonly referred to as *hydraulic diffusivity*.

Since the 1950s, several mathematical expressions for hydraulic diffusivity have been developed as more information became available. **Hayami (1951)** developed the first expression for hydraulic diffusivity. He was followed by **Dooge (1973)** and later by **Dooge and others (1982)**, culminating in the work of **Ponce (1991)**, who identified the Vedernikov number in the expression for *dynamic* hydraulic diffusivity. The question remains as to whether the aforementioned improvements are indeed such when practical applications are considered. This article reviews and compares Hayami's and Ponce's expressions for hydraulic diffusivity. In the process, we manage to advertently cast a doubt on the existential nature of dynamic waves (**Ponce, 2023a**).

2. KINEMATIC HYDRAULIC DIFFUSIVITY

Hayami (1951) pioneered the diffusion wave approach with free-surface flow (**Ponce, 2023a**) with the specific purpose for flood routing applications. His approach has been widely referred to in the literature as Hayami's *diffusion analogy*. He neglected the inertia terms in the equation of motion and combined the resulting equation with the equation of water continuity to derive a second-order equation with discharge as dependent variable. The coefficient of the second-order term, i.e., Hayami's hydraulic diffusivity, is (**Ponce, 2014**):

$$\nu = \frac{q_o}{2S_o} \quad (1)$$

in which ν = hydraulic diffusivity, q_o = reference unit-width discharge, and S_o = bottom slope.

Since Hayami's formulation explicitly **excludes** inertia, it is properly a *kinematic* hydraulic diffusivity, to follow the work of **Lighthill and Whitham (1955)** on kinematic wave theory. Therefore:

$$\nu_k = \frac{q_o}{\dots} \quad (2)$$

$$\frac{q_o}{2S_o}$$

in which $v_k = \textit{kinematic}$ hydraulic diffusivity.

3. DYNAMIC HYDRAULIC DIFFUSIVITY

Using linear theory, **Dooge (1973)** derived the complete convection-diffusion equation of flood flows, **including** the inertia terms, leading to the formulation of a *dynamic* hydraulic diffusivity:

$$v_d = \frac{q_o}{2S_o} \left(1 - \frac{F^2}{4} \right) \quad (3)$$

in which $v_d = \textit{dynamic}$ hydraulic diffusivity. In Equation 3, $F = \textit{Froude}$ number of the equilibrium flow, defined as follows: $F = v / (g D)^{1/2}$, in which $v = \textit{mean flow velocity}$, $g = \textit{gravitational acceleration}$, and $D = \textit{hydraulic depth}$, defined as $D = A / T$, in which $A = \textit{flow area}$, and $T = \textit{channel top width}$.

Dooge and others (1982) improved the original Dooge formulation (Eq. 3) by expressing the fraction within parenthesis in terms of the exponent β of the discharge-flow area rating $Q = \alpha A^\beta$, to yield:

$$v_d = \frac{q_o}{2S_o} [1 - (\beta - 1)^2 F^2] \quad (4)$$

In fact, for Chezy friction in a hydraulically wide channel, $\beta = 1.5$. Therefore, for this case, Eq. 4 is shown to be equivalent to Eq. 3.

Ponce (**1991a**; **1991b**) recognized the term $(\beta - 1)F$ in Eq. 4 as the Vedernikov number V (**Ponce, 2014b**), thereby reducing the expression for dynamic hydraulic diffusivity to the following:

$$v_d = \frac{q_o}{2S_o} (1 - V^2) \quad (5)$$

We note that while the kinematic hydraulic diffusivity (Eq. 2) is independent of the Vedernikov number, the dynamic hydraulic diffusivity (Eq. 5) has a clearly defined threshold for $V = 1$, being positive for $V < 1$, leading to wave attenuation, and negative for $V > 1$, leading to wave amplification.

4. TYPES OF WAVES IN OPEN-CHANNEL FLOW

While Equation 2 does not include the inertia terms, Equation 5 clearly does. Therefore, Eq. 5 should be a more accurate description of hydraulic diffusivity. To throw additional light on the issue, Table 1 shows the various types of waves in unsteady open-channel flow, including classical and common names.

Equation 2 was derived by Hayami by **neglecting inertia** in the equation of motion (Table 1, Line 2), i.e., a kinematic wave with diffusion; therefore, we confirm the appropriateness of the name *kinematic hydraulic diffusivity*. On the other hand, Eq. 5 was derived by **considering all terms** in the equation of motion (Table 1, Line 4), i.e., a mixed kinematic-dynamic wave, commonly referred to as a **dynamic wave**; therefore, the name *dynamic hydraulic diffusivity* appears justified at this juncture.

Table 1. Types of waves in unsteady open-channel flow.

Line No.	Classical wave name	Terms included in the equation of motion					Common name
		Local inertia	Convective inertia	Pressure gradient	Friction	Gravity	
1	Kinematic				✓	✓	Kinematic
2	Kinematic with diffusion			✓	✓	✓	Diffusion
3	Dynamic	✓	✓	✓			"Gravity"
4	Mixed kinematic-dynamic	✓	✓	✓	✓	✓	"Dynamic"

We have confirmed that Eq. 2 is an approximation for hydraulic diffusivity and that Eq. 5 is the complete expression. The question arises as to what is the actual difference between the two formulations when practical applications are considered. In the following section (Section 5) we evaluate the difference between the two alternative formulations for hydraulic diffusivity (Eqs. 2 and 5) using a catchment model of overland flow.

5. COMPARISON BETWEEN DIFFUSIVITIES

In this section, we use an overland flow model to test the difference between the two formulations of hydraulic diffusivity, Eqs. 2 and 5. The model simulates catchment dynamics using an open-book geometric configuration, i.e., two planes (left and right) and one channel in the middle (Fig. 1) (Ponce, 1986). The model's rainfall-runoff results are compared for the following two conditions: (1) kinematic hydraulic diffusivity (Eq. 2); and (2) dynamic hydraulic diffusivity (Eq. 5) (Ponce and Scott, 2022).

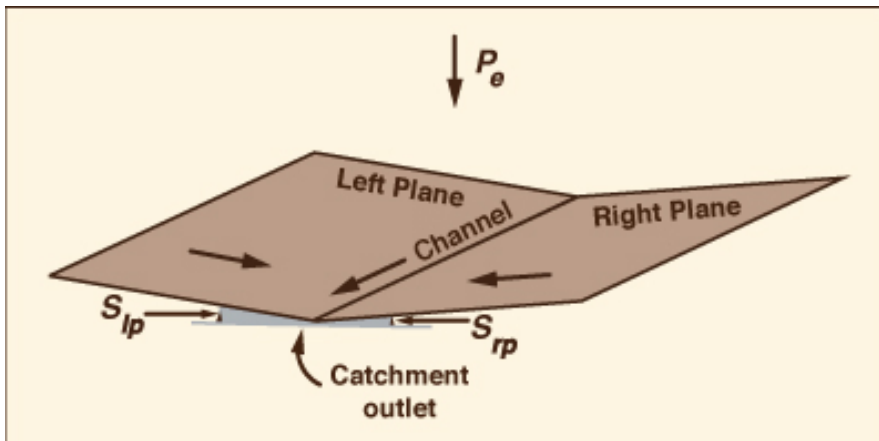


Fig. 1 ONLINEOVERLAND open-book geometric configuration.

The results show negligible differences between the two diffusivities under a wide range of flow conditions. Kinematic waves (Table 1, Line 1) and diffusion waves (Table 1, Line 2) were used in the tests because these were the only ones likely to be encountered in actual practice. The Vedernikov number in the planes was quite low, around 0.03.

Figure 2 shows the outflow hydrographs for drainage areas varying from 18 hectares (small) to 576 hectares (midsize). Figure 3 shows the outflow hydrographs for plane slopes (left and right) varying from 0.01 (steep) to 0.00001 (mild). No appreciable differences are observed between hydrographs generated using kinematic [(a) figures] and dynamic [(b) figures] hydraulic diffusivities.

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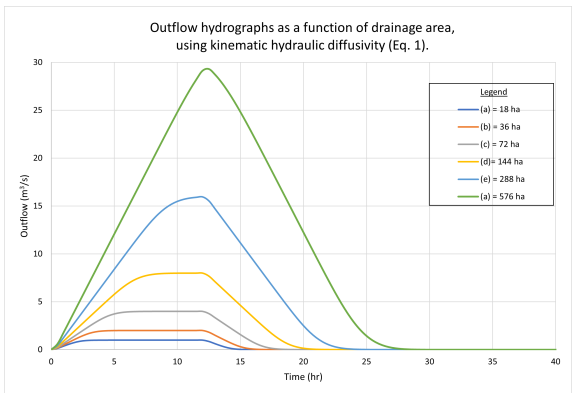


Fig. 2 (a) Outflow hydrographs as a function of drainage area using kinematic hydraulic diffusivity.

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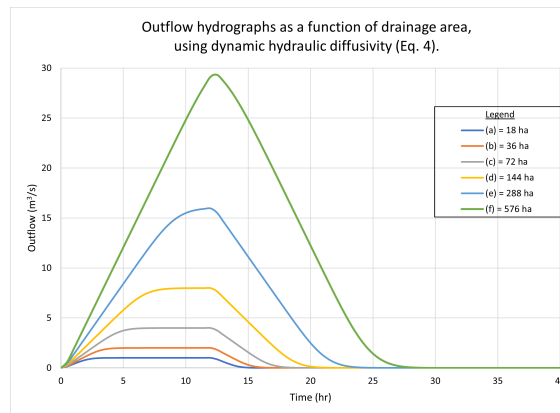


Fig. 2 (b) Outflow hydrographs as a function of drainage area using dynamic hydraulic diffusivity.

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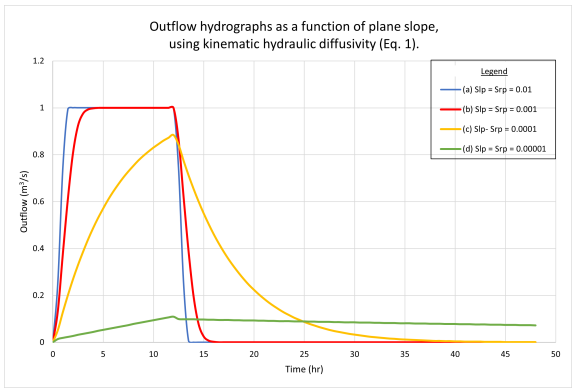


Fig. 3 (a) Outflow hydrographs as a function of plane slope using kinematic hydraulic diffusivity.

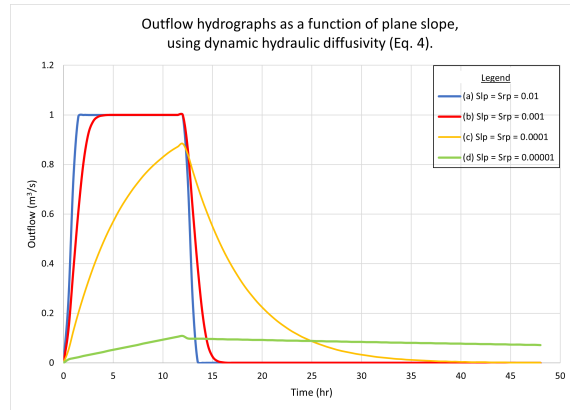


Fig. 3 (b) Outflow hydrographs as a function of plane slope using dynamic hydraulic diffusivity.

The lack of any appreciable difference between output hydrographs when comparing results using kinematic and dynamic diffusivities (Eqs. 2 and 5) is attributed to the type of modeling used in this study. Catchment flow typically features low Vedernikov numbers, particularly in the planes; in this case, $V \simeq 0.03$. In fact, for $V = 0.03$: $\nu_d = 0.9991 \nu_k$.

The examples shown in Figs 2 and 3 depict kinematic waves (Table 1, Line 1) and diffusion waves (Table 1, Line 2), since these are the ones likely to be encountered in actual practice. By definition, these waves are not subject to substantial diffusion; thus, featuring relative small values of either of the two diffusivities. We conclude that when modeling catchment flows using kinematic and diffusion waves, the difference between hydraulic diffusivities is likely to be negligible.

6. DOES THE DYNAMIC WAVE REALLY EXIST IN PRACTICE?

A question to be addressed at this juncture is the nature of mixed kinematic-dynamic waves (Table 1, Line 4), the so-called "dynamic" waves of hydraulic engineering (Fread, 1985). Do the dynamic waves really exist in practice? It is appropriate to begin this discussion with a quote from **Lighthill and Whitham (1955)**, who, in their seminal treatise on kinematic waves, saw fit to describe the dynamic waves as follows.

"... In some applications, including the case of flood waves, kinematic waves and dynamic waves are both possible together. However, the dynamic waves have both a much higher wave velocity and also a rapid attenuation. Hence, although any disturbance sends some signal downstream at the ordinary wave velocity for long gravity waves, this signal is too weak to be noticed at any considerable distance downstream, and the main signal arrives in the form of a kinematic wave at a much slower speed." (Page 285).

Further confirmation of the correctness of Lighthill and Whitham's observations regarding the strong dissipative nature of dynamic waves was given by **Ponce and Simons (1977)**, who used linear stability analysis to derive celerity and attenuation functions for the four types of shallow-water waves shown in Table 1. Ponce and Simons (1977) identified the dimensionless wavenumber σ , as the fundamental

physical parameter describing the propagation of shallow-water waves. Their findings are summarized in Fig. 4.

The amount of wave diffusion is expressed in terms of the *logarithmic decrement* δ , a measure of the rate at which the wave changes upon propagation (Wylie, 1966). The definition of logarithmic decrement is: $\delta = \ln Q_1 - \ln Q_0$, or, alternatively: $Q_1 = Q_0 e^{\delta}$, in which Q_0 = flood discharge at the start of the measurement, and Q_1 = flood discharge after an elapsed time equal to one (sinusoidal) period of propagation.

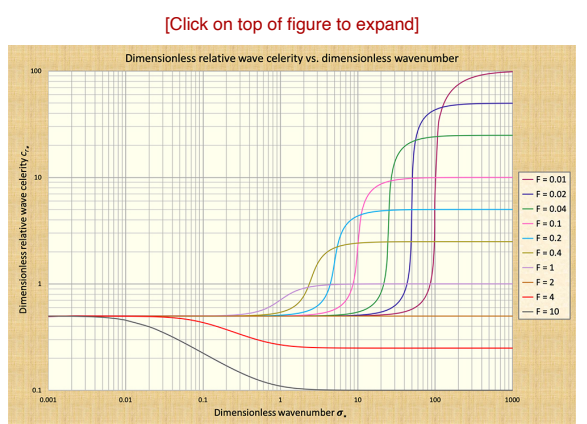


Fig. 4 (a) Dimensionless relative celerity vs dimensionless wavenumber in unsteady open-channel flow.

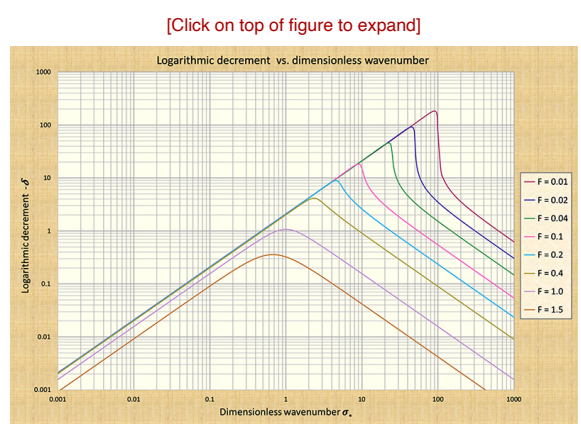


Fig. 4 (b) Logarithmic decrement vs dimensionless wavenumber in unsteady open-channel flow.

The findings of Ponce and Simons (1977) confirm the very strong dissipative tendencies of mixed kinematic-dynamic waves, commonly referred to in the hydraulic engineering literature simply as "dynamic waves" (Fread, 1985). The peak attenuation amount occurs at the point of inflection in the dimensionless relative celerity c_{r*} vs dimensionless wavenumber σ_* [compare Fig. 4 (a) with Fig. 4 (b)].

Note that the amount of wave attenuation increases markedly with a decrease in equilibrium flow Froude number. For the lowest Froude number shown in Fig. 4, $F = 0.01$, the peak logarithmic decrement is $\delta = 180$, corresponding to $\sigma_* = 90$ [Fig. 4 (b)].

Table 2 shows peak logarithmic decrements (Column 3) for the range of subcritical Froude numbers between 0.01 and 1.5 (Column 1) shown in Fig. 4 (b) and their corresponding wave attenuation amounts (Column 5). The prevailing wave type is shown to be mixed kinematic-dynamic for all subcritical flows, and kinematic with diffusion for critical and supercritical flows (Column 6). The amount of wave attenuation, shown at or very close to 1, indicates that the entire range of subcritical flows is subject to very strong dissipative tendencies. This result further casts doubts on the mixed kinematic-dynamic wave as a basis for computation. The mixed kinematic-dynamic wave is simply not there for us to calculate!

Table 2. Peak logarithmic decrements of the mixed kinematic-dynamic wave model for Froude numbers between F = 0.01 and F = 1.5.					
[1]	[2]	[3]	[4]	[5]	[6]

Froude No.	Dimensionless wavenumber σ_*	Peak logarithmic decrement δ	e^{δ}	Wave attenuation $(1 - e^{-\delta})$	Wave type
0.01	90	180	0.	1.	Mixed kinematic-dynamic
0.02	45	95	0.	1.	Mixed kinematic-dynamic
0.04	23	48	0.	1.	Mixed kinematic-dynamic
0.1	9	19	0.	1.	Mixed kinematic-dynamic
0.2	4.5	9.5	0.	1.	Mixed kinematic-dynamic
0.4	2.3	4.2	0.015	0.985	Mixed kinematic-dynamic
1.0	1.1	1.1	0.333	0.667	Kinematic with diffusion
1.5	0.7	0.37	0.691	0.309	Kinematic with diffusion

Having demonstrated that the mixed kinematic-dynamic wave may not be there for us, we still have a choice of using or not the dynamic hydraulic diffusivity (Eq. 5) in **all** applications of unsteady free-surface flow, including both catchment and channel flow. Since Eq. 5 is indeed the complete equation to model diffusivity and it does not significantly complicate the calculation of runoff diffusion it may still be regarded as the best choice.

7. SUMMARY

A comparison between kinematic and dynamic hydraulic diffusivities using an actual catchment model of overland flow is accomplished. The kinematic hydraulic diffusivity is the well known Hayami diffusivity, which is independent of the Vedernikov number. On the other hand, the dynamic hydraulic diffusivity is a function of the Vedernikov number. Several examples of rainfall-runoff conversions using the catchment model show that the difference between these two formulations of hydraulic diffusivity is negligible. This may be attributed to the very low Vedernikov numbers typically featured in catchment models, or to the specification of kinematic and diffusion waves in the test cases, since these waves are the only ones likely to be encountered in actual practice.

The question of the true nature of mixed kinematic-dynamic waves is examined using available analytical data. It is concluded that given their extremely strong dissipative tendencies, mixed waves may not be there for us to calculate. Nevertheless, the dynamic hydraulic diffusivity is advocated as the method of choice because it is the complete solution, it is applicable to all types of routing, including catchment and channel flow, and it does not significantly complicate the methodology.

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